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Group Delay in Rectangular Waveguides

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Abstract

The purpose of this analysis is to calculate the effect that dispersiveness in a waveguide has on modulated signals and how to possibly compensate for certain changes from system to system. For the present, the compensation of interest is the preservation of second-order, or parabolic effects, in modulated signals when lengths of waveguide runs are changed. Changes in modulation propagate down a waveguide at the so-called "group velocity". In this paper, we will first review the expression governing the propagation of signals in a waveguide. Then we will review the expression for group velocity. Following that, we will derive an expression for the slope of the group delay time, a quantity that is related to the distortion of modulated signals as they travel down a waveguide. Finally, a method for compensation is suggested for preservation of the transfer characteristic of a waveguide system, as other considerations require that its length be changed.

1. Propagation Characteristics of Waveguides

Propagation of a single frequency signal in a uniform rectangular waveguide is governed by the expression:

$$\beta = \frac{1}{c}\sqrt{\omega^2 - \omega_c^2} \qquad \text{Equation 1}$$

In this expression, c is the velocity of light in the waveguide medium, which for air is:

$$c = 2.998 \text{X} 10^8 \text{ meters/second.} \qquad \text{Equation 2}$$

The quantity β is the so-called propagation constant. It is a measure of the phase shift a signal undergoes as it travels down a waveguide. In the MKS rationalized system of units, the units of β are radians/meter. We shall come back to the significance of β in a moment. Let us now turn our attention to ω and ω_c, which are generally more familiar than β. These are "angular" frequencies given by:

$$\omega = 2\pi f \qquad \text{Equation 3}$$

and

$$\omega_c = 2\pi f_c . \qquad \text{Equation 4}$$

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where f is the signal frequency (in Herz) and f_c is the cutoff frequency of the waveguide mode (in Herz), which in rectangular waveguide is usually the dominant TE_{10} mode. For this mode, the cut-off frequency is simply given by

$$f_c = \frac{c}{2a} \qquad \text{Equation 5}$$

where a is the waveguide width (in meters).

Now turning our attention back to Eq. 1, and substituting Eqs. 2 and 3 for the angular frequencies, we obtain the expression for the propagation constant,

$$\beta = \frac{2\pi}{c}\sqrt{f^2 - f_c^2} \qquad \text{Equation 6}$$

This expression will be key in determining signal distortion later in this paper. Before proceeding, it is interesting to point out the significance of the propagation constant, β. It expresses the axial spatial variation of waves as they propagate down a waveguide. For example, in the TE_{10} mode, the electric field in the waveguide in space and time is given by:

$$E_y = A\sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta z - \psi) \qquad \text{Equation 7}$$

This is based on a rectangular coordinate system shown in Fig. 1.

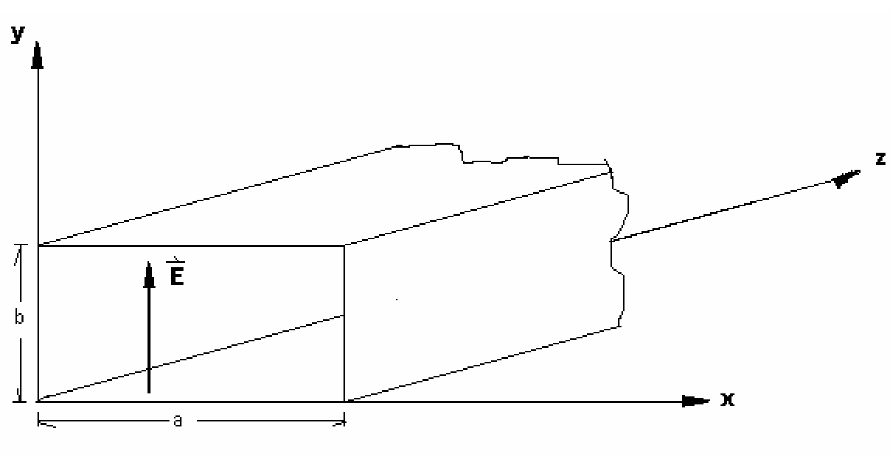

Figure 1. Rectangular Waveguide and Coordinate System

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In Eq. 7, it is evident that β expresses the phase shift per unit length; ψ is an arbitrary phase angle depending upon the onset of time measurement. Another way of looking at β is that for a waveguide length L, the total phase shift experienced by a field, or by a signal, is:

$$\theta = \beta L \qquad \text{Equation 8}$$

This phase shift is in radians and is expressed in degrees as:

$$\theta = \frac{360^0}{2\pi}\beta L \qquad \text{Equation 9}$$

Yet another way to look at the propagation constant, β, is to consider that the guide wavelength, λ_g, is related to β, by:

$$\beta = \frac{2\pi}{\lambda_g} \qquad \text{Equation 10}$$

The discussion beyond Eq. 6 above has been academic and has been provided to give an intuitive feel for the propagation constant, β. For the problem of interest, namely the second-order, or parabolic, effects on a modulated signal propagating down a waveguide, only Eq. 6 is important.

2. Group Velocity

The changes in modulation on a signal as it travels down a waveguide is related to the "group velocity", v_g. The reason for our dwelling on the propagation constant, β, above has been that the group velocity is directly related to it through the expression:

$$v_g = \frac{d\omega}{d\beta} \qquad \text{Equation 11}$$

We now proceed to derive the expression for group velocity from Eq. 11 and Eq. 6. Because of the form of Eq. 11, it is evident that Eq. 6 has to be inverted. Presently, Eq. 6 expresses β as a function of ω; to find the derivative expressed by Eq. 11, we need ω as a function of β. To do so, we can manipulate Eq. 6 algebraically to obtain:

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$$\omega^2 = c^2\beta^2 + \omega_c^2 \qquad \text{Equation 12}$$

If we now take the derivative of each side of Eq. 11with respect to β, we obtain:

$$2\omega\frac{d\omega}{d\beta} = 2c^2\beta \qquad \text{Equation 13}$$

or

$$v_g = \frac{d\omega}{d\beta} = c^2\frac{\beta}{\omega} \qquad \text{Equation 14}$$

Substituting Eqs. 1, 2 and 3 into Eq. 14, we obtain the expression for group velocity:

$$v_g = c\sqrt{1-\left(\frac{f_c}{f}\right)^2} \qquad \text{Equation 15}$$

It is to be noted that in a waveguide, the operating frequency, f, is always greater than the cut-off frequency, f_c. Hence, the group velocity is always less than the speed of light. The group velocity approaches the velocity of light as frequency increases above cut-off. In the special case of coaxial waveguides, the cut-off frequency is zero and the group velocity is constant with frequency and equal to the speed of light.

But in rectangular waveguides, the group velocity is a function of frequency and not a constant. This leads to the distortion of a modulated signal as it propagates down a waveguide. The next section of this paper will provide a measure for this distortion.

3. Phase Velocity

The phase velocity is the velocity at which a field component propagates. In a waveguide, it is different from the group velocity. In a coaxial line it is equal to the group velocity. The phase velocity, v_p, is given by the expression:

$$v_p = \frac{\omega}{\beta} \qquad \text{Equation 16}$$

By combining Eqs. 1, 2 and 4 with Eq. 16, we obtain the expression for phase velocity:

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$$v_p = \frac{c}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad \text{Equation 17}$$

Whereas the group velocity, v_g, was less than the speed of light, it is evident that the phase velocity, v_p, is greater than the speed of light since the operating frequency f is always greater than the cut-off frequency f_c. It is interesting to note from Eqs. 15 and 17,

$$v_g v_p = c^2 \qquad \text{Equation 18}$$

It is to borne in mind that the phase velocity is the velocity at which a field component, such as the electric field, travels down the guide, while the group velocity is the velocity at which energy travels down the guide.

4. Group Delay, Dispersion and Signal Distortion

The expressions for propagation, group velocity and phase velocity were discussed above. From these expressions, it is seen that the group and phase velocities vary with frequency. In a coaxial line or in free-space wave, the group and phase velocities are constant with frequency, and both are equal to the speed of light. But in waveguides, they vary with frequency, which means that a modulated signal, which can be thought of as a spectrum of frequencies, would undergo distortion, or more accurately, dispersion, as it travels down a waveguide. In general, a waveguide can be regarded as a dispersive transmission line, whereas, a coaxial line or free space are regarded as non-dispersive.

We will now derive an expression that is a measure of the dispersion a modulated signal would undergo as it travels down a waveguide of length L. Toward this end, we consider the group delay, t_g, which is the time that it takes for a change in signal to propagate. The group delay is simply length divided by speed, or

$$t_g = \frac{L}{v_g} \qquad \text{Equation 19}$$

If we have a signal with a center frequency of f_0 and a bandwidth Δf, and Δf is small in comparison with f_0, we would expect that the variation in group delay, Δt_g , over the band would be given by

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$$\Delta t_g = \left.\frac{dt_g}{df}\right|_{f=f_0} \Delta f \qquad \text{Equation 20}$$

Substituting the expression for group velocity from Eq. 15 and taking the derivative of each side of Eq. 19, we obtain

$$\frac{dt_g}{df} = -\frac{L}{c}\frac{f_c^2}{f^3}\left[1-\left(\frac{f_c}{f}\right)^2\right]^{-\frac{3}{2}} \qquad \text{Equation 21}$$

This expression is basically the slope of the group delay. The group delay is directly proportional to the waveguide length and has a somewhat involved algebraic relationship to the operating frequency and the cut-off frequency. The higher above cut-off one operates, the less the dispersion. For a coaxial line, the cutoff frequency is zero and so is the slope of the group delay. For a coaxial line, the group velocity is non-dispersive, *i.e.*, constant. Note that the slope of the group delay is negative. This is it is decreasing as frequency increases. This is a direct consequence of group velocity increasing as frequency increases. Note that the slope is negative. This is because the group delay is decreasing as frequency increases. This is a direct consequence of group velocity increasing as frequency increases.

To obtain the variation in group delay for a signal of center frequency f_0 and a bandwidth Δf, one plugs in the appropriate values in Eqs. 20 and 21.

5. Circuit Behavior

We will now examine how the various quantities we have been discussing are related to the circuit behavior of a waveguide. We will concentrate on Eq. 9, which describes phase shift in terms familiar in circuit analysis. Our interest is generally in modulated signals and these are usually narrow band, *i.e.*, we have signals with center frequency f_0 and a bandwidth Δf, where

$$\Delta f << f_0 \qquad \text{Equation 22}$$

A Taylor Series Expansion of Eq. 9 takes the form:

$$\theta = \theta(f_0) + \left.\frac{d\theta}{df}\right|_{f=f_0} \Delta f + \frac{1}{2!}\left.\frac{d^2\theta}{df^2}\right|_{f=f_0} (\Delta f)^2 + \ldots\ldots\ldots\ldots \qquad \text{Equation 23}$$

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The initial term, $\theta(f_0)$, is the phase shift at the center of the band and is given by

$$\theta(f_0) = 360^0 \frac{f_0 L}{c} \sqrt{1 - \left(\frac{f_c}{f_0}\right)^2} \qquad \text{Equation 24}$$

The first order term is given by

$$\left.\frac{d\theta}{df}\right|_{f=f_0} = 360^0 \frac{L}{c} \left[1 - \left(\frac{f_c}{f_0}\right)^2\right]^{-\frac{1}{2}} \qquad \text{Equation 25}$$

which is obtained by taking the first derivative of Eq. 9. The second order term is given by

$$\left.\frac{d^2\theta}{df^2}\right|_{f=f_0} = -360^0 \frac{L}{c} \frac{f_c^2}{f_0^3} \left[1 - \left(\frac{f_c}{f_0}\right)^2\right]^{-\frac{3}{2}} \qquad \text{Equation 26}$$

which is the second derivative of Eq. 9, *i.e.*, the first derivative of Eq. 25. So, to second order, the change in phase over a band centered at f_0, with a bandwidth of Δf, becomes:

$$\Delta\theta = 360^0 \frac{L}{c} \left\{ \left[1 - \left(\frac{f_c}{f_0}\right)^2\right]^{-\frac{1}{2}} \Delta f + \frac{1}{2} \frac{f_c^2}{f_0^3} \left[1 - \left(\frac{f_c}{f_0}\right)^2\right]^{-\frac{3}{2}} (\Delta f)^2 \right\} \qquad \text{Equation 27}$$

To lend this expression more physical meaning rather than just mathematical meaning, it is instructive to substitute Eq. 15 and 19 into the first-order term and Eq. 21 into the second order term. This results in the following expression:

$$\Delta\theta = 360^0 t_g \Delta f + 180^0 \frac{dt_g}{df} (\Delta f)^2 \qquad \text{Equation 28}$$

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The first-order, or linear term, term is directly proportional to the group delay time, and the second-order, or parabolic, term is proportional to the slope of the delay time.

The key quantity for the parabolic behavior is evidently the first derivative of the group delay, what we have called the time delay slope. This has appeared when we considered a somewhat intuitive treatment of group velocity and group delay and made the leap that the time delay slope would be related to preserving the character of a modulated signal as it propagates down a waveguide. This slope is proportional to the length, L, of a waveguide of interest. It will be useful to define a new quantity, τ, as the group delay for a unit length of waveguide. We will use 1 meter as the unit length, since we are working in the MKS Rationalized System of Units, and since we wish to avoid any confusion or errors due to unit conversions. With τ being the group delay per unit length, we will use the term τ' to express its derivative with frequency. So, we have:

$$\tau' = \frac{d\tau}{df} \qquad \text{Equation 29}$$

Accordingly, Eq. 28 becomes:

$$\Delta\theta = 360^0 L\left\{\tau\Delta f + \frac{1}{2}\tau'(\Delta f)^2\right\} \qquad \text{Equation 29}$$

6. <u>Tables of Calculations</u>

Based on the above analysis, tables have been prepared in the form of a Microsoft Excel worksheet. The file, named “Group Delay.xls”, provides calculations of group velocity v_g, unit delay time, τ, and unit delay time slope, τ'. These parameters can be calculated for standard millimeter waveguide sizes by entering the WR designation and the operating frequency. An additional part to the table has been included so that these parameters may also be obtained for non-standard or oversized waveguides.

7. <u>Consideration of a Specific Problem</u>

Consider the specific problem of a waveguide system with an equalizer that has been specifically designed to compensate for the parabolic behavior of the length of waveguide incorporated. The equalizer likely represents a considerable investment in comparison to the waveguide run. If system changes dictate that the length of the waveguide run be changed, then it would be desirable to somehow preserve the parabolic behavior of the waveguide system in order to utilize the existing equalizer. Since the

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parabolic behavior is related to the cut-off frequency of the waveguide, it is conceivable that a change in waveguide size over part of the run could preserve the overall behavior.

a) Let L_a be the original waveguide length and L_b be the new waveguide length.
b) Let f_{c1} be the cut-off frequency of the original waveguide and f_{c2} be the cutoff frequency of a larger waveguide size to be spliced into the total run to increase it.
c) Let L_1 be the new length of waveguide with cut-off f_{c1} and L_2 be the new length of waveguide with cut-off frequency f_{c2}.
d) It is clear that $L_2 = L_b - L_1$
e) Let τ_1' be the unit delay time slope for the waveguide section with cut-off frequency f_{c1}, and τ_2' be that for the waveguide section of cut-off frequency, f_{c2}.
f) To preserve the parabolic behavior from the first system to the next, we must have the overall derivative $\frac{dt_g}{df}$ preserved, which implies that:

$$\tau_1' L_a = \tau_1' L_1 + \tau_2'(L_b - L_1) \qquad \text{Equation 28}$$

or

$$L_1 = \frac{\tau_2' L_b - \tau_1' L_a}{(\tau_2' - \tau_1')} \qquad \text{Equation 29}$$

Assuming:
a) A center frequency of 60 GHz,
b) A length of 17' for La,
c) A length of 21' for Lb,
d) A waveguide size of WR-15 for La and L1, and
e) A waveguide size of WR-19 for for L2,
We obtain from Eq. 29 and the Excel spreadsheet:

$$L_1 = \frac{.02458 \times 21 - .05885 \times 17}{(.02458 - .05885)} \qquad \text{Equation 30}$$

From which it follows that

$$L_1 = \frac{.5162 - 1.0004}{-.03427} = 14.13' \qquad \text{Equation 31}$$

and

$$L_2 = 21 - 14.13 = 6.87' \qquad \text{Equation 32}$$

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Thus, if the system length were to be be increased from 17' to 21', the way to preserve the parabolic behavior would be to subtract 3' of WR-15 waveguide and add in a 7' section of WR-19 waveguide.