Impossibility of 45-Degree Hybrid

Nicholas Estes, University of Notre Dame

March 2020

All matched, lossless, passive four-ports result in directional couplers that essentially route the signal from a given input port to two of the three possible output ports, with either a 90° or 180° phase shift between the outputs (quadrature and 180° couplers, respectively). It is of interest to ask if it is possible to implement a 45° hybrid in the same way. The typical restrictions on the hybrid are as follows:

- All ports are matched.
- The power from each individual input port is delivered to the output ports in an equal amplitude split and a 45° difference between the ports.
- The four-port is passive and its s-parameter matrix is unitary (lossless).
- The matrix is reciprocal such that $\mathbf{S}_{nm} = \mathbf{S}_{mn}$, $\forall\, m, n$

The proof as to why the above restrictions make a 45° hybrid impossible is well-covered in [Pozar(2012)], chapter 7. Here, the feasibility of a 45° hybrid considered for a looser set of restrictions, that is:

1. Only the input ports (which will be designated ports 2 and 3) are necessarily matched ($\mathbf{S}_{22} = \mathbf{S}_{33} = 0$).
2. The power from each individual input port is delivered totally to the output ports in an equal amplitude split and a 45° difference between the ports. A corollary is that the input ports are isolated from one another ($\mathbf{S}_{32} = \mathbf{S}_{23} = 0$).
3. The four-port is passive and its s-parameter matrix is unitary (lossless) ($\mathbf{S}\mathbf{S}^H = \mathbf{I}$), where $\mathbf{I}$ denotes the identity matrix and $\mathbf{S}^H$ is the conjugate transpose of the four-port.

The restrictions have been relaxed such that the matrix is no longer necessarily reciprocal, which is to say the hybrid perhaps uses relatively exotic materials. Additionally, the input ports are the only ports that need be matched *ab initio*. It just needs to be demonstrated that the above restrictions introduce a contradiction.

Begin by applying condition 1 to a general four-port S-parameter matrix:

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & 0 & S_{23} & S_{24} \\ S_{31} & S_{32} & 0 & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \tag{1}$$

Now, following condition 2:

$$\begin{aligned} |S_{12}|^2 + |S_{42}|^2 &= 1 \\ |S_{12}|^2 = |S_{42}|^2 &= 0.5 \\ |S_{12}| = |S_{42}| &= \frac{1}{\sqrt{2}} \end{aligned} \tag{2}$$

and likewise:

$$|S_{13}| = |S_{43}| = \frac{1}{\sqrt{2}} \tag{3}$$

Arbitrarily, port 1 is chosen as the 'through' port for an input from port 2, and port 4 is the corresponding 45° port. Conversely, port 4 and port 1 are the through and 45° ports for input from port 3, respectively. Therefore:

$$\begin{aligned} S_{12} = S_{43} &= \frac{1}{\sqrt{2}} \\ S_{42} = S_{13} &= \frac{1+j}{2} \end{aligned} \tag{4}$$

as $\frac{1+j}{\sqrt{2}}$ is the unit vector at 45° in the complex plane. Substitute the above relations into $\mathbf{S}$, and set S_{23} and S_{32} to 0, as all power is assumed to go to the output ports:

$$\mathbf{S} = \begin{bmatrix} S_{11} & \frac{1}{\sqrt{2}} & \frac{1+j}{2} & S_{14} \\ S_{21} & 0 & 0 & S_{24} \\ S_{31} & 0 & 0 & S_{34} \\ S_{41} & \frac{1+j}{2} & \frac{1}{\sqrt{2}} & S_{44} \end{bmatrix} \tag{5}$$

Now it is possible to explicitly multiply out S and its complex conjugate S^H and equate the resulting matrix to I.

$$\mathbf{SS}^H = \begin{bmatrix} S_{11} & \frac{1}{\sqrt{2}} & \frac{1+j}{2} & S_{14} \\ S_{21} & 0 & 0 & S_{24} \\ S_{31} & 0 & 0 & S_{34} \\ S_{41} & \frac{1+j}{2} & \frac{1}{\sqrt{2}} & S_{44} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{21}^* & S_{31}^* & S_{41}^* \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1-j}{2} \\ \frac{1-j}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ S_{14}^* & S_{24}^* & S_{34}^* & S_{44}^* \end{bmatrix} = \mathbf{I} \tag{6}$$

For clarity's sake, the identity matrix is printed below:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The matrix multiplication can be evaluated term-by-term and terms may be simplified if possible. Beginning with the first element:

$$\mathbf{SS}^H_{11} = |S_{11}|^2 + 0.5 + 0.5 + |S_{14}|^2 = 1 = \mathbf{I}_{11} \tag{8}$$

and by inspection:

$$S_{11} = S_{14} = 0 \tag{9}$$

Analogously, the same procedure can be performed for $\mathbf{SS}^H_{44}$ to find:

$$S_{41} = S_{44} = 0 \tag{10}$$

which simplifies the matrix multiplication further, to:

$$\mathbf{SS}^H = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1+j}{2} & 0 \\ S_{21} & 0 & 0 & S_{24} \\ S_{31} & 0 & 0 & S_{34} \\ 0 & \frac{1+j}{2} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & S^*_{21} & S^*_{31} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1-j}{2} \\ \frac{1-j}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & S^*_{24} & S^*_{34} & 0 \end{bmatrix} \tag{11}$$

Ultimately, even though neither isolation between the output ports nor matched output ports were assumed, these conditions are required given the relatively short set of initial restrictions. As $\mathbf{SS}^H_{12}$ and $\mathbf{SS}^H_{13}$ are now trivially equal to 0 (and thus automatically satisfy the equality to $\mathbf{I}_{12}$ and $\mathbf{I}_{13}$) the next relevant term is $\mathbf{SS}^H_{14}$, where:

$$\begin{aligned} \frac{1}{\sqrt{2}}\frac{1-j}{2} + \frac{1}{\sqrt{2}}\frac{1+j}{2} &= 0 \\ \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}} &= 0 \end{aligned} \tag{12}$$

Having found a contradiction, it is now proven that, even given the relaxed requirements on the four-port, a *perfect* 45° hybrid is not realizable. Possible modifications to the network would be to allow loss (non-unitary behavior), using a non-passive network, or reducing the isolation requirements.

References

[Pozar(2012)] D. M. Pozar. *Microwave engineering; 4th ed.* Wiley, Hoboken, NJ, 2012.